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Chapman–Enskog method for a phonon gas with finite heat flux

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Abstract

The Chapman–Enskog perturbation method for a phonon gas is investigated with the use of Callaway's model for the Boltzmann–Peierls equation. Assuming that the effective relaxation time for normal processes is small and the effective relaxation time for resistive processes is large, this perturbation method proposes to expand the phase density about a displaced Planck distribution and to include the above two relaxation times in the expansion. The main advantage of using the displaced Planck distribution is that the drift velocity of a phonon gas is incorporated into the model in a non-perturbative manner. The result is a system of nonlinear second-order parabolic equations for the energy density and the drift velocity which, unlike the usual set of hydrodynamic equations, does not restrict the magnitude of the individual components of the drift velocity and the heat flux in any way. This system is linearly stable at all wavelengths and is also fully consistent with the second law of thermodynamics in the sense that there exists a macroscopic entropy density which depends locally on the hydrodynamic variables and satisfies the balance equation with a non-negative entropy production due to both resistive and normal processes.

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1. Introduction

Heat transfer and second sound in dielectrics with large drift velocities of a phonon gas have been investigated by Nielsen and Shklovskii [1] (see also [2]). Using the Boltzmann–Peierls equation [3, 4] and the displaced Planck distribution [5, 6], these authors derived nonlinear hydrodynamic equations which describe the motion of a phonon gas at low temperatures. The derivation consists in substituting the displaced Planck distribution into the system of equations for the energy density and the heat flux obtained from the Boltzmann–Peierls equation. Since the displaced Planck distribution is a highly nonlinear function of the drift velocity, this approach permits the inclusion of the drift velocity in a non-perturbative manner and allows arbitrarily large values for the individual components of the heat flux. It yields a closed set of nonlinear differential equations with the independent hydrodynamic variables being the temperature and the drift velocity. The equations of Nielsen and Shklovskii contain the source term due to resistive processes and they seem particularly suited for describing phonon flows in the regime where the effective relaxation time for normal processes (τ_N) is much smaller than the effective relaxation time for resistive processes (τ_R). In this regime, during the first time period, normal processes make the phonon gas approach the displaced Planck distribution, and then during the longer time period, resistive processes return it to the equilibrium Planck distribution. Larecki and Piekarski [7] showed that the Nielsen–Shklovskii evolution system is symmetric hyperbolic in variables equivalent to the temperature and the drift velocity. Next, for the phonon gas with nondispersion and isotropy for frequency spectrum, Larecki [8] derived the explicit equations of first-order symmetric hyperbolic form for another set of variables, namely the energy density and the components of the heat flux.

In a sense, the hydrodynamic model of Nielsen–Shklovskii type can be regarded as a zeroth-order approximation in the Chapman–Enskog perturbation scheme. The traditional Chapman–Enskog method of solution of phonon kinetic equations [9–13] is based on an expansion about a local Planck distribution and, as such, requires that the phonon gas be close to local thermodynamic equilibrium. Assuming that τ_N is small and τ_R is large, the present paper develops systematically a modification of the traditional method so as to allow for an expansion about the displaced Planck distribution. This modification is effectively realized by expanding the phase density in powers of τ_N and $1/\tau_R$ and by including the spatial derivatives of the relevant hydrodynamic variables in the expansion. To zeroth order the expansion yields the hyperbolic equations for the energy density and the drift velocity. These equations are very similar to the equations derived by Nielsen and Shklovskii, the only difference being that the drift velocity balance equation does not contain the source term due to resistive processes. However, this source term is recovered when the first-order approximation is considered. At first order, the primary result is a set of nonlinear second-order parabolic equations (governing the evolution of the same variables) which, unlike the usual set of parabolic equations for a phonon gas, does not restrict the magnitude of the individual elements of the drift velocity and the heat flux in any way. It is significant that our model involves the two relaxation times. The first relaxation time (τ_R) appears in the source term on the right-hand side of an equation for the drift velocity, whereas the second relaxation time (τ_N) appears in the expression for the deviatoric part of the flux of the heat flux. This expression is linear in the energy density, nonlinear in the drift velocity, and linear in the spatial derivatives of these hydrodynamic variables. We illustrate the method by demonstrating that the model is linearly stable at all wavelengths. If τ_R equals infinity, the reference background solution to the evolution system is characterized by arbitrary constant values of the energy density and the drift velocity.

The first-order approximation is fully consistent with the second law of thermodynamics because it enables us to define a macroscopic entropy density: this entropy density depends locally on the hydrodynamic variables and satisfies the balance equation with a non-negative entropy production due to both resistive and normal processes. The calculations involved in extending the Chapman–Enskog expansion scheme to the second- and higher-order corrections can lead to the hydrodynamic equations not fully consistent with the second law of thermodynamics and since the first-order approximation is usually considered an adequate description of a phonon gas, results above first order will not be presented.

We employ units which are defined by setting $\hbar = k_B = 1$. For the sake of simplicity, we use the Boltzmann–Peierls equation with Callaway's collisional terms [14, 15]. The effective relaxation times are assumed to be the constant quantities. No distinction is made between longitudinal and transverse phonons. The dispersion relation for all types of phonons has the form $\Omega = c|\mathbf{p}|$, where c is the constant Debey speed. We let the components $(p^i) = (p^1, \ldots, p^n)$ of the momentum[1] $\mathbf{p}$ range from $-\infty$ to $+\infty$. As to the value of n in $(p^1, \ldots, p^n)$, we wish to introduce and discuss the two- and three-dimensional phonon models, treating them in a uniform way throughout the whole discussion. Consequently, we consider the spacetime whose spatial dimension is either two or three; hence $n = 2$ or $n = 3$.

This paper is organized as follows. Section 2 formulates the Chapman–Enskog expansion of the phase density about a displaced Planck distribution. Section 3 is devoted to the study of the first-order approximation. Section 4 shows consistency between the hydrodynamic equations derived from this approximation and the second law of thermodynamics. Section 5 proves linear stability of the model. Section 6 gives final remarks.

2. Expansion about a displaced Planck distribution

2.1. Callaway's model

We consider a gas composed of phonons. The fundamental object for describing this gas is the distribution function f that depends on the position $\mathbf{x} \in \mathbb{R}^n$ and the momentum $\mathbf{p} \in \mathbb{R}^n$ and whose evolution is governed by a kinetic equation of the form

$$\partial_t f + cg^i \partial_i f = J_R(f) + J_N(f), \tag{2.1}$$

where (g^i) are the components of $\mathbf{g} := \mathbf{p}/|\mathbf{p}|$, (∂_t, ∂_i) denote differentiation with respect to (t, x^i) and $J_R(f)$, $J_N(f)$ stand for the collision terms due to resistive and normal processes, respectively. The exact kinetic-theory expressions for $J_R(f)$, $J_N(f)$ are quite formidable. For our present purposes it will be enough to restrict attention to a particularly simple kinetic theory which nevertheless gives realistic results that can be compared to experiments. The kinetic theory used is the kinetic model of Callaway [14], which is a relaxation-type model where the collision terms $J_R(f)$, $J_N(f)$ are approximated as

$$J_R(f) = \frac{1}{\tau_R}(F - f), \qquad J_N(f) = \frac{1}{\tau_N}(F_d - f). \tag{2.2}$$

Here (τ_R, τ_N) are the effective relaxation times, which we assume are the constant quantities, and (F, F_d) are the equilibrium and displaced Planck distributions defined by

$$F := \frac{y}{\exp(cp/T) - 1}, \qquad F_d := \frac{y}{\exp[(cp/T_d)(1 - \mathbf{v}\cdot\mathbf{g})] - 1}, \tag{2.3}$$

where

$$y := n(2\pi)^{-n}, \qquad p := |\mathbf{p}|, \qquad |\mathbf{v}| < 1.$$

Note that y specifies the smallest element of the phase space that can accommodate a phonon. The scalar functions $T = T(t, \mathbf{x})$, $T_d = T_d(t, \mathbf{x})$ represent two different temperature fields and the vector function $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ represents the drift velocity of a phonon gas. We refer to $\mathbf{v}$ as the drift velocity, even though the true expression for this velocity is obtained by multiplying $\mathbf{v}$ by c. Clearly, the temperature and velocity fields are not arbitrary. They must be taken so as to give agreement with the basic properties of the collision operators in the Boltzmann–Peierls equation [3, 4]. Consequently, since the relaxation times (τ_R, τ_N) are assumed not to depend on $\mathbf{p}$, we fix T, T_d and $\mathbf{v}$ in such a way that F reproduces correctly the actual energy density e and F_d reproduces correctly the actual energy density e and the actual heat flux $\mathbf{q}$:

[1] Since $\hbar = 1$, this momentum can also be interpreted as the phonon wave vector.

$$c\int pF\,\mathrm{d}^n\mathbf{p} = c\int pF_d\,\mathrm{d}^n\mathbf{p} = e := c\int pf\,\mathrm{d}^n\mathbf{p}, \tag{2.4a}$$

$$c^2\int p\mathbf{g}F_d\,\mathrm{d}^n\mathbf{p} = \mathbf{q} := c^2\int p\mathbf{g}f\,\mathrm{d}^n\mathbf{p}. \tag{2.4b}$$

Equations (2.4) are consistent with the fact that resistive processes conserve energy and normal processes conserve energy and momentum:

$$\int pJ_R(f)\,\mathrm{d}^n\mathbf{p} = 0, \qquad \int pJ_N(f)\,\mathrm{d}^n\mathbf{p} = 0, \qquad \int p\mathbf{g}J_N(f)\,\mathrm{d}^n\mathbf{p} = \mathbf{0}.$$

We verify that $|\mathbf{q}| < ce$. Indeed, for any two momenta $\mathbf{p}$ and $\mathbf{p}'$, we have the inequality $\mathbf{p}\cdot\mathbf{p}' \leqslant |\mathbf{p}||\mathbf{p}'|$. Multiplying this inequality by $c^4 f(t,\mathbf{x},\mathbf{p})f(t,\mathbf{x},\mathbf{p}')$ and integrating over $(\mathbf{p},\mathbf{p}')$ yields $|\mathbf{q}|^2 < c^2e^2$. Equations (2.4) make it possible to relate T to e and $(T_d,\mathbf{v})$ to $(e,\mathbf{q})$. Explicitly, we obtain

$$T = \left(\frac{e}{n\chi}\right)^{\frac{1}{n+1}}, \qquad T_d = \left[\frac{e}{(n+u)\chi}\right]^{\frac{1}{n+1}}(1-u)^{\frac{n+3}{2(n+1)}}, \tag{2.5a}$$

$$\mathbf{v} = \frac{2n\mathbf{q}}{(n+1)ce + \sqrt{(n+1)^2c^2e^2 - 4n|\mathbf{q}|^2}}, \tag{2.5b}$$

where[2]

$$\chi := \frac{2(n-1)\pi y}{nc^n}\int_0^\infty \frac{z^n}{\exp(z)-1}\,\mathrm{d}z, \quad u := \frac{n[(n+1)ce - \sqrt{(n+1)^2c^2e^2 - 4n|\mathbf{q}|^2}]}{(n+1)ce + \sqrt{(n+1)^2c^2e^2 - 4n|\mathbf{q}|^2}}. \tag{2.6}$$

Using (2.5b) and (2.6), it is easily confirmed that

$$0 \leqslant u = |\mathbf{v}|^2 < 1, \qquad \mathbf{q} = \frac{(n+1)ce}{n+u}\mathbf{v}. \tag{2.7}$$

The passage from $(e,\mathbf{q})$ to $(T_d,\mathbf{v})$ or $(e,\mathbf{v})$ is a diffeomorphic change of variables and $\mathbf{q}$ vanishes if $\mathbf{v} = \mathbf{0}$.

The flux of the heat flux is given by

$$\mathbb{M}^{ij} := c^3\int pg^ig^jf\,\mathrm{d}^n\mathbf{p}. \tag{2.8}$$

From (2.8) we see that the deviatoric part of $\mathbb{M}^{ij}$ is expressible as

$$M^{ij} := c^3\int pg^{\langle i}g^{j\rangle}f\,\mathrm{d}^n\mathbf{p},$$

where angle brackets denote the symmetric traceless part, e.g.,

$$g^{\langle i}g^{j\rangle} := g^ig^j - \frac{1}{n}\delta^{ij}, \qquad g^{\langle i}g^jg^{k\rangle} := g^ig^jg^k - \frac{3}{n+2}g^{(i}\delta^{jk)}.$$

Round brackets indicate symmetrization and δ^{ij} stands for the Kronecker delta. On the understanding that (q^i) are the components of $\mathbf{q}$, a set of equations for (e,q^i) emerging from (2.1) and (2.2) is of the form

$$\partial_t e + \partial_i q^i = 0, \tag{2.9a}$$

[2] Since $y = n(2\pi)^{-n}$, it follows from (2.6) that $\chi = \zeta(3)/(\pi c^2)$ if $n = 2$ and $\chi = \pi^2/(30c^3)$ if $n = 3$. As to the meaning of ζ, this is the Riemannian ζ-function.

$$\partial_t q^i + \frac{c^2}{n}\delta^{ij}\partial_j e + \partial_j M^{ij} = -\frac{1}{\tau_R} q^i. \tag{2.9b}$$

System (2.9) is not a determined system. The problem is that, because there appears the second-order moment flux M^{ij} in (2.9b), an additional relation must be supplied to close system (2.9). We use a variant of the Chapman–Enskog method of solution of Callaway's model in order to solve this problem.

2.2. *A formal abstract structure of the method*

In order to allow the relaxation time τ_N to become small and the relaxation time τ_R to become large, it is convenient to introduce the small parameter ε into Callaway's model as follows:

$$\partial_t f + cg^i \partial_i f = \frac{\varepsilon}{\tau_R}(F - f) + \frac{1}{\varepsilon\tau_N}(F_d - f), \tag{2.10a}$$

$$\partial_t e + \partial_i q^i = 0, \tag{2.10b}$$

$$\partial_t q^i + \frac{c^2}{n}\delta^{ij}\partial_j e + \partial_j M^{ij} = -\frac{\varepsilon}{\tau_R} q^i. \tag{2.10c}$$

With the help of (2.10), we can speak of the regime where normal processes substantially dominate over resistive ones. Proceeding formally, we adopt the expansion

$$f = \sum_{l=0}^{\infty} \varepsilon^l f_l \tag{2.11}$$

and subsequently assume that every f_l in (2.11) depends on (t, x^i) through the hydrodynamic variables (e, q^i) and their spatial derivatives up to order l:

$$f_l = f_l\big(e, \partial_{\mathbf{x}} e, \ldots, \partial_{\mathbf{x}}^l e; q^i, \partial_{\mathbf{x}} q^i, \ldots, \partial_{\mathbf{x}}^l q^i; \mathbf{p}\big). \tag{2.12}$$

The small parameter ε is set equal to one ($\varepsilon = 1$) in the ultimate asymptotic expression for f. Expansion (2.11) generates a similar expansion for the moment flux M^{ij} in equation (2.10c), so that we can write M^{ij} in the form

$$M^{ij} = \sum_{l=0}^{\infty} \varepsilon^l M_l^{ij}, \tag{2.13}$$

where

$$M_l^{ij} := c^3 \int p g^{\langle i} g^{j\rangle} f_l \, \mathrm{d}^n \mathbf{p}.$$

The zeroth-order distribution f_0 is required to give the exact hydrodynamic variables via

$$e = c\int p f_0 \, \mathrm{d}^n\mathbf{p}, \qquad q^i := c^2 \int p g^i f_0 \, \mathrm{d}^n \mathbf{p}, \tag{2.14}$$

while the higher-order terms are required to satisfy the so-called compatibility conditions

$$e_l := c\int p f_l \, \mathrm{d}^n \mathbf{p} = 0, \qquad q_l^i := c^2 \int p g^i f_l \, \mathrm{d}^n \mathbf{p} = 0 \quad (l \geqslant 1). \tag{2.15}$$

Because of (2.14) and (2.15), each order of the expansion of f yields the same energy density and heat flux, i.e., the hydrodynamic variables (e, q^i) are not effectively expanded in powers of ε.

The time derivative is likewise expressed as a power series in ε. Precisely speaking, since equation (2.10c) contains the small parameter and the energy and heat flux balance equations link space and time derivatives, it is necessary to introduce an expansion of the time derivatives of (e, q^i) as

$$\partial_t e = \sum_{l=0}^{\infty} \varepsilon^l E_l, \qquad \partial_t q^i = \sum_{l=0}^{\infty} \varepsilon^l Q_l^i. \tag{2.16}$$

The precise meaning of the expansion coefficients in (2.16) is obtained by using equations (2.10*b*), (2.10*c*), (2.13) and (2.15) which lead to

$$E_0 = -\partial_j q^j, \qquad E_l = 0 \quad (l \geqslant 1), \tag{2.17a}$$

$$Q_0^i = -\frac{c^2}{n}\delta^{ij}\partial_j e - \partial_j M_0^{ij}, \tag{2.17b}$$

$$Q_1^i = -\partial_j M_1^{ij} - \frac{1}{\tau_R} q^i, \qquad Q_l^i = -\partial_j M_l^{ij} \quad (l \geqslant 2). \tag{2.17c}$$

Now, for each non-negative integer r,

$$\partial_t \partial_{\mathbf{x}}^r e = \partial_{\mathbf{x}}^r \partial_t e, \qquad \partial_t \partial_{\mathbf{x}}^r q^i = \partial_{\mathbf{x}}^r \partial_t q^i.$$

In view of (2.11), (2.12) and (2.16), we then verify by direct calculation that

$$\partial_t f = \sum_{l=0}^{\infty} \varepsilon^l \mathcal{F}_l, \tag{2.18}$$

where

$$\mathcal{F}_l := \sum_{m=0}^{l} \sum_{r=0}^{m} \left[\frac{\partial f_m}{\partial\left(\partial_{\mathbf{x}}^r e\right)} \partial_{\mathbf{x}}^r E_{l-m} + \frac{\partial f_m}{\partial\left(\partial_{\mathbf{x}}^r q^i\right)} \partial_{\mathbf{x}}^r Q_{l-m}^i \right].$$

Inserting (2.11) and (2.18) into (2.10*a*) and equating the coefficients of equal powers of ε gives

$$f_0 = F_d, \qquad f_1 = -\tau_N[\mathcal{F}_0 + cg^i \partial_i f_0], \tag{2.19a}$$

$$f_2 = -\tau_N \left[\mathcal{F}_1 + cg^i \partial_i f_1 + \frac{1}{\tau_R}(f_0 - F) \right], \tag{2.19b}$$

$$f_{l+1} = -\tau_N \left[\mathcal{F}_l + cg^i \partial_i f_l + \frac{1}{\tau_R} f_{l-1} \right] \quad (l \geqslant 2). \tag{2.19c}$$

This is a set of recurrent equations in which the explicit expressions for $\partial_i f_l$ are provided by

$$\partial_i f_l = \sum_{m=0}^{l} \left[\frac{\partial f_l}{\partial\left(\partial_{\mathbf{x}}^m e\right)} \partial_i \left(\partial_{\mathbf{x}}^m e\right) + \frac{\partial f_l}{\partial\left(\partial_{\mathbf{x}}^m q^j\right)} \partial_i \left(\partial_{\mathbf{x}}^m q^j\right) \right] \quad (l \geqslant 0)$$

and in which f_0 is identified with the displaced Planck distribution.

With the aid of (2.16), we see that the lowest-order $O(\varepsilon^0)$ balance equations are

$$\partial_t e = E_0, \qquad \partial_t q^i = Q_0^i.$$

Utilizing (2.17*a*) and (2.17*b*), these equations can also be brought into the form

$$\partial_t e + \partial_i q^i = 0, \tag{2.20a}$$

$$\partial_t q^i + \frac{c^2}{n}\delta^{ij}\partial_j e + \partial_j M_0^{ij} = 0, \tag{2.20b}$$

where

$$M_0^{ij} := c^3 \int p g^{\langle i} g^{j\rangle} f_0 \, \mathrm{d}^n\mathbf{p} = c^3 \int p g^{\langle i} g^{j\rangle} F_d \, \mathrm{d}^n\mathbf{p}. \tag{2.21}$$

The result of using equations (2.3), (2.5)–(2.7) and carrying out the **p**-integration in (2.21) is

$$M_0^{ij} = \frac{(n+1)c^2 e}{n+u} v^{\langle i} v^{j\rangle} = \frac{2nc}{(n+1)ce + \sqrt{(n+1)^2c^2e^2 - 4n|\mathbf{q}|^2}} q^{\langle i} q^{j\rangle}. \tag{2.22}$$

From (2.20) and (2.22) we then obtain a system of $n+1$ hyperbolic differential equations for (e, q^i). This system is consistent with the nonlinear model of Nielsen and Shklovskii [1] and reduces to the evolution system derived by Larecki [8] if $n = 3$. The only difference is that the heat flux balance equation does not contain the source term. However, this source term is recovered when the first-order approximation is concerned; see equation (3.6*b*).

3. First-order Chapman–Enskog

In order to find an explicit expression for f_1, we must first calculate $\partial_i f_0$ and $\mathcal{F}_0$ and subsequently use (2.19*a*). Since $f_0 = F_d$ and F_d is a function of (t, x^i) only in its dependence on (e, q^i), we have

$$\begin{aligned}
\partial_i f_0 &= \frac{\partial F_d}{\partial e}\partial_i e + \frac{\partial F_d}{\partial q^j}\partial_i q^j, \\
\mathcal{F}_0 &= \frac{\partial F_d}{\partial e} E_0 + \frac{\partial F_d}{\partial q^i} Q_0^i \\
&= -\frac{\partial F_d}{\partial e}\partial_i q^i - \frac{c^2}{n}\delta^{ij}\frac{\partial F_d}{\partial q^i}\partial_j e - \frac{\partial F_d}{\partial q^i}\partial_j M_0^{ij}.
\end{aligned}$$

From (2.3), (2.5)–(2.7) and (2.22) it will be possible to evaluate $\partial_i f_0$ and $\mathcal{F}_0$ directly in terms of $(T_d, \mathbf{v})$ and the spatial derivatives of these hydrodynamic variables. Also, given (2.19*a*) and the above observations, it will be possible to relate f_1 to $(T_d, \mathbf{v})$ and the spatial derivatives of $(T_d, \mathbf{v})$. A rather tedious calculation yields the result

$$f_1 = -\tau_N \frac{c^2 p}{T_d} F_d(1 + \bar{F}_d)\vartheta_{ij}\left[g^i(g^j - 2v^j) + \frac{1}{n-u}(n + 1 - 2\mathbf{v}\cdot\mathbf{g})v^i v^j\right], \tag{3.1}$$

where (v^i) are the components of $\mathbf{v}$ and

$$\bar{F}_d := \frac{1}{y}F_d, \qquad \vartheta_{ij} := \partial_{\langle i} v_{j\rangle} - \frac{1}{T_d} v_{\langle i}\partial_{j\rangle} T_d.$$

Here and throughout this paper, we adopt the useful convention whereby the indices are lowered and raised with the Kronecker deltas, e.g.,

$$v_i := \delta_{ij} v^j, \qquad \vartheta^{ij} := \delta^{ik}\delta^{jl}\vartheta_{kl}.$$

Moreover, as noted earlier, $u = |\mathbf{v}|^2$ and angle brackets denote the symmetric traceless part. If equation (2.5*a*) for T_d is used, then ϑ_{ij} appears in the form

$$\vartheta_{ij} = \partial_{\langle i} v_{j\rangle} - \frac{1}{(n+1)e} v_{\langle i}\partial_{j\rangle} e + \frac{n+2+u}{2(1-u)(n+u)} v_{\langle i}\partial_{j\rangle} u.$$

We have verified that expression (3.1) satisfies the compatibility conditions

$$e_1 := c\int p f_1 \, \mathrm{d}^n\mathbf{p} = 0, \qquad q_1^i := c^2 \int p g^i f_1 \, \mathrm{d}^n\mathbf{p} = 0.$$

The calculations yielding these conditions are quite complicated, however.

Now, it is straightforward but computationally somewhat cumbersome to show that the substitution of (3.1) into the right-hand side of the equation

$$N^{ij} := M_1^{ij} = c^3 \int p g^{\langle i} g^{j\rangle} f_1 \, \mathrm{d}^n \mathbf{p}$$

leads to the following formula for N^{ij}:

$$N^{ij} = -\frac{nc^3 e\tau_N}{n+u}[A\mu^{ij} + Bv_k\mu^{k\langle i}v^{j\rangle} + C\mu_{kl}v^k v^l v^{\langle i}v^{j\rangle}], \tag{3.2}$$

where

$$\mu_{ij} := (1-u)\vartheta_{ij} = (1-u)\left[\partial_{\langle i}v_{j\rangle} - \frac{1}{(n+1)e}v_{\langle i}\partial_{j\rangle}e\right] + \frac{n+2+u}{2(n+u)}v_{\langle i}\partial_{j\rangle}u. \tag{3.3}$$

In (3.3), the Einstein summation rule, according to which a repeated index implies summation over all values of that index, is used. The coefficient A is a function of u and is defined by

$$A := \begin{cases} \dfrac{2}{u^2}\left[(1-u)\sqrt{1-u} + \dfrac{1}{2}(3u-2)\right] & \text{if} \quad n=2, \\ \dfrac{1}{u^2}\left[\dfrac{(1-u)^2}{2\sqrt{u}}\ln\left(\dfrac{1+\sqrt{u}}{1-\sqrt{u}}\right) + \dfrac{1}{3}(5u-3)\right] & \text{if} \quad n=3. \end{cases} \tag{3.4}$$

Concerning (B, C), these coefficients can be represented as algebraic functions of A and u:

$$B := \begin{cases} \dfrac{2}{u}(3-4A) & \text{if} \quad n=2, \\ \dfrac{2}{u}\left(\dfrac{8}{3}-5A\right) & \text{if} \quad n=3, \end{cases}$$
$$C := \begin{cases} \dfrac{6[2(2-u)A-3+u]}{u^2(2-u)} & \text{if} \quad n=2, \\ \dfrac{8(4u-21)+105(3-u)A}{6u^2(3-u)} & \text{if} \quad n=3. \end{cases} \tag{3.5}$$

In the limits $u \to 0_+$ and $u \to 1_-$, we have

$$\lim_{u\to 0_+} A = \begin{cases} \frac{3}{4} & \text{if} \quad n=2, \\ \frac{8}{15} & \text{if} \quad n=3, \end{cases} \qquad \lim_{u\to 1_-} A = \begin{cases} 1 & \text{if} \quad n=2, \\ \frac{2}{3} & \text{if} \quad n=3, \end{cases}$$
$$\lim_{u\to 0_+} B = \begin{cases} -1 & \text{if} \quad n=2, \\ -\frac{16}{21} & \text{if} \quad n=3, \end{cases} \qquad \lim_{u\to 1_-} B = \begin{cases} -2 & \text{if} \quad n=2, \\ -\frac{4}{3} & \text{if} \quad n=3, \end{cases}$$
$$\lim_{u\to 0_+} C = \begin{cases} -\frac{3}{16} & \text{if} \quad n=2, \\ 0 & \text{if} \quad n=3, \end{cases} \qquad \lim_{u\to 1_-} C = \begin{cases} 0 & \text{if} \quad n=2, \\ \frac{1}{3} & \text{if} \quad n=3. \end{cases}$$

Consequently, the coefficients (A, B, C) are well-behaved as functions of u near $u = 0$ and $u = 1$.

The balance equations, after truncating expansion (2.16) by retaining only the first two terms, read

$$\partial_t e + \partial_i q^i = 0, \tag{3.6a}$$

$$\partial_t q^i + \frac{c^2}{n}\delta^{ij}\partial_j e + \partial_j\left(M_0^{ij} + N^{ij}\right) = -\frac{1}{\tau_R}q^i. \tag{3.6b}$$

With the aid of (2.7) and (2.22), we can transform (3.6) to the system of equations for (e, v^i). This system written in full is

$$\partial_t e + \frac{c(n+1)}{n+u} v^j \partial_j e + \frac{ce(n+1)}{n+u} \partial_j v^j - \frac{ce(n+1)}{(n+u)^2} v^j \partial_j u = 0, \tag{3.7a}$$

$$\begin{aligned}\partial_t v^i &+ \frac{c(1-u)}{e(n+1)} \delta^{ij} \partial_j e - \frac{c(n-1)(1-u)}{e(n+1)(n-u)} v^i v^j \partial_j e \\ &- \frac{c(1-u)}{n-u} v^i \partial_j v^j + c v^j \partial_j v^i - \frac{c}{n+u} \delta^{ij} \partial_j u + \frac{c(n-1)}{n^2-u^2} v^i v^j \partial_j u \\ &+ \frac{n+u}{ce(n+1)} \partial_j N^{ij} + \frac{2(n+u)}{ce(n+1)(n-u)} v^i v_j \partial_k N^{jk} = -\frac{(n+u)v^i}{(n-u)\tau_R},\end{aligned} \tag{3.7b}$$

where N^{ij} is defined by (3.2).

To sum up, equations (3.7a) and (3.7b) were derived by using F_d as the base function and by approximating f with a truncated Chapman–Enskog-like expansion about F_d having the form

$$f = F_d + f_1. \tag{3.8}$$

Most conventional methods rely on perturbative expansions for the distribution functions in terms of a local Planckian, whereas approximation (3.8) relies on the displaced Planck distribution, namely F_d. The latter is fundamentally a nonequilibrium distribution function that includes the effects of non-zero drift velocity and heat flux in a non-perturbative manner. Precisely speaking, with the exception of the obvious inequalities $|\mathbf{v}| < 1$ and $|\mathbf{q}| < ce$, there are effectively no limitations on the values of $|\mathbf{v}|$ and $|\mathbf{q}|$, i.e., one can handle problems with large components[3] of the drift velocity and the heat flux. This is a definite improvement over traditional approaches which only make allowances for small deviations in the drift velocity and the heat flux from zero. Moreover, the obtained evolution system is expected to be able to describe weakly nonlocal phenomena because it involves the first and second spatial derivatives of the hydrodynamic variables.

4. Entropy balance law

System (3.7) has an entropy function

$$s := \frac{(n+1)(1-u)}{(n+u)} \frac{e}{T_d}. \tag{4.1}$$

This entropy function is identical to the hyperbolic entropy of the Nielsen–Shklovskii model [1, 8] and is obtained by substituting $f = F_d$ into the following kinetic-theory expression for the entropy functional [4]:

$$s(f) := y \int \left[\left(1 + \frac{f}{y}\right) \ln\left(1 + \frac{f}{y}\right) - \frac{f}{y} \ln\left(\frac{f}{y}\right)\right] \mathrm{d}^n\mathbf{p}.$$

From (2.5a) and (3.7) we find the entropy balance law to be

$$\partial_t s + \partial_i \varphi^i = \sigma, \tag{4.2}$$

where

$$\varphi^i := c s v^i - \frac{1}{cT_d} v_j N^{ij}, \qquad \sigma := \frac{(n+1)eu}{(n+u)T_d \tau_R} - \frac{1}{c} \partial_i \left(\frac{1}{T_d} v_j\right) N^{ij}. \tag{4.3}$$

For essentially obvious reasons, we call φ^i the entropy flux and σ the entropy production. The second law of thermodynamics, when applied to equation (4.2), requires that $\sigma \geqslant 0$. Since

[3] As demonstrated in [1], the conditions under which $|\mathbf{v}|$ is comparable to 1 and $|\mathbf{q}|$ is comparable to ce may be attained experimentally.

the first term on the right-hand side of equation (4.3)$_2$ for σ is strictly positive if $|\mathbf{v}| \neq 0$ and vanishes if $|\mathbf{v}| = 0$, the aim here is to consider only the problem of verifying whether or not the second term given by

$$\Sigma := -\frac{1}{c}\partial_i\left(\frac{1}{T_d}v_j\right)N^{ij}$$

is non-negative.

In order to arrive at the explicit form of Σ, we require some preliminary definitions:

$$\Theta := \frac{1}{e}w^k\partial_k e, \qquad \Gamma := \partial_k v^k, \qquad \Xi := w^k w^l \partial_{\langle k}v_{l\rangle},$$
$$\Theta_i := \frac{1}{e}(\partial_i e - w_i w^k \partial_k e), \qquad \Upsilon_i := \frac{1}{2}(w^k\partial_i v_k - w^k\partial_k v_i),$$
$$\Gamma_i := \frac{1}{2}(w^k\partial_i v_k + w^k\partial_k v_i - 2w_i w^k w^l \partial_k v_l),$$
$$\gamma_{ij} := \partial_{\langle i}v_{j\rangle} - w^k w_{\langle i}\partial_{j\rangle}v_k + \frac{1}{n-1}[\partial_k v^k + (n-2)w^k w^l \partial_k v_l]w_{\langle i}w_{j\rangle} - w_{\langle i}\delta^k_{j\rangle}w^l\partial_l v_k.$$

Here (w^i) are the components of a unit vector in the direction of $\mathbf{v}$, i.e.,

$$w^i := \frac{v^i}{|\mathbf{v}|}. \tag{4.4}$$

Using these definitions and equation (2.5a) for T_d, it can be checked by straightforward if very tedious working that

$$\Sigma = \frac{nc^2 e\tau_N}{(n+u)T_d}\left[\frac{H}{18n(n-1)(1-u)(n-u)(n+u)^2}\gamma^2 \right.$$
$$\left. + \frac{D}{2(n+1)(1-u)(n+u)^2}\gamma_i\gamma^i + (1-u)A\gamma_{ij}\gamma^{ij}\right], \tag{4.5}$$

where

$$\gamma := 6[n^2 + 2(n-1)u - u^2]\Xi + (n+1)(n+2+u)u\Gamma - n(1-u)(n+u)|\mathbf{v}|\Theta,$$
$$\gamma_i := (n+1)[2n + (4-n)u - u^2]\Gamma_i + (n+1)(n+2+u)u\Upsilon_i - (1-u)(n+u)|\mathbf{v}|\Theta_i$$

and

$$D := \begin{cases} 1 - A & \text{if} \quad n = 2, \\ \frac{2}{3} - A & \text{if} \quad n = 3, \end{cases} \qquad H := \begin{cases} \frac{3}{2}[(2-u)A - 1] & \text{if} \quad n = 2, \\ 3(3-u)A - 4 & \text{if} \quad n = 3. \end{cases} \tag{4.6}$$

In the case when $|\mathbf{v}|$ approaches zero and the values of w^i and $\partial_i v_j$ are arbitrarily fixed, equation (4.5) simplifies to

$$\Sigma = \begin{cases} \dfrac{3c^2 e\tau_N}{4T_d}\alpha_{ij}\alpha^{ij} \geqslant 0 & \text{if} \quad n = 2, \\ \dfrac{8c^2 e\tau_N}{15T_d}\alpha_{ij}\alpha^{ij} \geqslant 0 & \text{if} \quad n = 3, \end{cases}$$

where $\alpha_{ij} := \partial_{\langle i}v_{j\rangle}$. For very large values of $|\mathbf{v}|$, since

$$\lim_{u\to 1_-}(1-u)A = 0, \qquad \lim_{u\to 1_-}\frac{D}{1-u} = \begin{cases} 1 & \text{if} \quad n = 2, \\ \frac{1}{3} & \text{if} \quad n = 3, \end{cases} \qquad \lim_{u\to 1_-}\frac{H}{1-u} = 0,$$

we obtain approximately

$$\Sigma \cong \begin{cases} \dfrac{c^2 e\tau_N}{81T_d}\gamma_i\gamma^i & \text{if} \quad n = 2, \\ \dfrac{c^2 e\tau_N}{512T_d}\gamma_i\gamma^i & \text{if} \quad n = 3. \end{cases}$$

In the general case, from (3.4) and (4.6) we find

$$A = \begin{cases} \dfrac{1 + 2\sqrt{1-u}}{(1+\sqrt{1-u})^2} & \text{if} \quad n = 2, \\ \dfrac{(1-u)^2}{2}\displaystyle\int_{-1}^{1} \dfrac{(1-z^2)^2}{(1-\sqrt{u}z)^5}\,\mathrm{d}z & \text{if} \quad n = 3, \end{cases}$$

$$D = \begin{cases} \dfrac{1-u}{(1+\sqrt{1-u})^2} & \text{if} \quad n = 2, \\ \dfrac{(1-u)^2}{2}\displaystyle\int_{-1}^{1} \dfrac{(1-z^2)(z-\sqrt{u})^2}{(1-\sqrt{u}z)^5}\,\mathrm{d}z & \text{if} \quad n = 3, \end{cases}$$

$$H = \begin{cases} \dfrac{3(1-u)\sqrt{1-u}}{(1+\sqrt{1-u})^2} & \text{if} \quad n = 2, \\ \dfrac{3(1-u)^2}{2(3-u)}\displaystyle\int_{-1}^{1} \dfrac{[(3-u)z^2 - 4\sqrt{u}z + 3u - 1]^2}{(1-\sqrt{u}z)^5}\,\mathrm{d}z & \text{if} \quad n = 3. \end{cases}$$

Combined with $0 \leqslant u < 1$, this shows that

$$A > 0, \qquad D > 0, \qquad H > 0. \tag{4.7}$$

By using (4.5) and (4.7), we deduce the inequality $\Sigma \geqslant 0$ with equality if and only if $\gamma = 0$, $\gamma_i = 0$ and $\gamma_{ij} = 0$. Consequently, the entropy production is non-negative,

$$\sigma = \frac{(n+1)eu}{(n+u)\tau_R} + \Sigma \geqslant 0,$$

and our model is verified to satisfy the second law of thermodynamics. Conditions (4.7) are also relevant to the question of linear stability of system (3.7); see section 5.

We finally mention the following. Equation (4.1) defines an entropy function of the hyperbolic–parabolic system (3.7) in the sense of Kawashima and Shizuta [16] or Kawashima and Yong [17]. Employing the results of these authors, one can prove that this entropy function is linked with the existence of a symmetric form of the system.

5. Linear plane-wave perturbations and stability

In this section, we derive the dispersion relations for linear plane-wave perturbations about a constant background equilibrium state of a phonon gas. The difference between the actual value of a field W at a given point of spacetime and the value which W has in the background equilibrium state will be denoted by δW. The quantities δe and δv^i are the fields which describe the perturbations of a phonon gas about its equilibrium state. Any fields which do not include the prefix δ refer to the background configuration, and are assumed to satisfy equations (3.7*a*) and (3.7*b*), in addition to being constant. The equations of motion for the perturbation fields δW are obtained by linearizing system (3.7) about the equilibrium state.

If $\tau_R \neq \infty$, the background drift velocity is zero and the equations are

$$\partial_t \delta e + \frac{ce(n+1)}{n}\partial_j \delta v^j = 0, \tag{5.1a}$$

$$\partial_t \delta v^i + \frac{c}{e(n+1)}\delta^{ij}\partial_j \delta e + \frac{n}{ce(n+1)}\partial_j \delta N^{ij} = -\frac{1}{\tau_R}\delta v^i, \tag{5.1b}$$

where

$$\delta N^{ij} := \begin{cases} -\frac{3}{4}c^3 e\tau_N \delta^{k\langle i}\partial_k \delta v^{j\rangle} & \text{if} \quad n=2, \\ -\frac{8}{15}c^3 e\tau_N \delta^{k\langle i}\partial_k \delta v^{j\rangle} & \text{if} \quad n=3. \end{cases} \tag{5.2}$$

If $\tau_R = \infty$, the background drift velocity is arbitrary and the equations for δe and δv^i can be written as

$$\partial_t \delta e + \frac{c(n+1)}{n+u}v^j\partial_j \delta e + \frac{ce(n+1)}{n+u}\partial_j \delta v^j - \frac{2ce(n+1)}{(n+u)^2}v_k v^j \partial_j \delta v^k = 0, \tag{5.3a}$$

$$\begin{aligned} \partial_t \delta v^i &+ \frac{c(1-u)}{e(n+1)}\delta^{ij}\partial_j \delta e - \frac{c(n-1)(1-u)}{e(n+1)(n-u)}v^i v^j \partial_j \delta e \\ &- \frac{c(1-u)}{n-u}v^i \partial_j \delta v^j + cv^j \partial_j \delta v^i - \frac{2c}{n+u}\delta^{ij}v_k \partial_j \delta v^k + \frac{2c(n-1)}{n^2-u^2}v^i v^j v_k \partial_j \delta v^k \\ &+ \frac{n+u}{ce(n+1)}\partial_j \delta N^{ij} + \frac{2(n+u)}{ce(n+1)(n-u)}v^i v_j \partial_k \delta N^{jk} = 0. \end{aligned} \tag{5.3b}$$

Here δN^{ij} is defined by

$$\delta N^{ij} := -\frac{nc^3 e\tau_N}{n+u}[A\delta\mu^{ij} + Bv_k v^{\langle i}\delta\mu^{j\rangle k} + Cv^{\langle i}v^{j\rangle}v^k v^l \delta\mu_{kl}], \tag{5.4}$$

where

$$\delta\mu^{ij} := (1-u)\left[\delta^{k\langle i}\partial_k \delta v^{j\rangle} - \frac{1}{(n+1)e}v^{\langle i}\delta^{j\rangle k}\partial_k \delta e\right] + \frac{n+2+u}{n+u}v_l v^{\langle i}\delta^{j\rangle k}\partial_k \delta v^l. \tag{5.5}$$

As to the definitions of the coefficients in (5.4), see equations (3.4) and (3.5).

We consider plane-wave solutions of the form

$$\delta e = \delta X \exp[\mathrm{i}(\omega t - k_l x^l)], \qquad \delta v^j = \delta X^j \exp[\mathrm{i}(\omega t - k_l x^l)], \tag{5.6}$$

where (k_l) are the components of the wave vector $\mathbf{k}$ and $(\delta X, \delta X^j)$ define the complex constant amplitude of the wave. As usual, we have chosen to write equations (5.6) with a frequency, which is defined so that its real part gives the oscillation frequency of the mode and its imaginary part gives the decay (or growth) time scale of the mode. Under these circumstances, the perturbation equations become a set of algebraic equations:

$$U^\alpha_\beta(\omega, \mathbf{k})\delta X^\beta = 0, \tag{5.7}$$

where $U^\alpha_\beta(\omega, \mathbf{k})$ is a $(n+1)\times(n+1)$ complex-valued matrix and $\delta X^\beta := (\delta X, \delta X^j)$ represents the list of the $n+1$ perturbation fields. The index β runs over these $n+1$ fields, while the index α runs over the $n+1$ equations governing the perturbation variables. There exist plane-wave solutions of equations (5.7) whenever ω and (k_l) have values which satisfy the condition

$$\mathfrak{A} := \det[U^\alpha_\beta(\omega, \mathbf{k})] = 0. \tag{5.8}$$

The resulting relation between ω and $\mathbf{k}$ is called the dispersion relation.

In the case $\tau_R \neq \infty$, it follows directly from (5.1), (5.2) and (5.6) that

$$\varpi\delta X - \frac{ek(n+1)}{n}\xi_l \delta X^l = 0, \tag{5.9a}$$

$$\frac{k}{n+1}\delta X\xi^j - e(\varpi - \mathrm{i}\mathfrak{d})\delta X^j + \frac{\mathrm{i}k^2 e}{2(n+1)}[n\delta X^j + (n-2)\xi_l \delta X^l \xi^j] = 0, \tag{5.9b}$$

where

$$\xi^j := \frac{k^j}{|\mathbf{k}|} \tag{5.10}$$

and

$$\varpi := \begin{cases} \dfrac{3}{4}\tau_N\omega & \text{if } n = 2, \\ \dfrac{8}{15}\tau_N\omega & \text{if } n = 3, \end{cases} \qquad k := \begin{cases} \dfrac{3}{4}c\tau_N|\mathbf{k}| & \text{if } n = 2, \\ \dfrac{8}{15}c\tau_N|\mathbf{k}| & \text{if } n = 3, \end{cases} \qquad \mathfrak{d} := \begin{cases} \dfrac{3\tau_N}{4\tau_R} & \text{if } n = 2, \\ \dfrac{8\tau_N}{15\tau_R} & \text{if } n = 3. \end{cases}$$

The introduction of the notation

$$\mathfrak{A}_1 := 2(n+1)\varpi - [nk^2 + 2(n+1)\mathfrak{d}]\mathrm{i},$$
$$\mathfrak{A}_2 := n(n+1)\varpi^2 - n[(n-1)k^2 + (n+1)\mathfrak{d}]\mathrm{i}\varpi - (n+1)k^2$$

enables condition (5.8) to be written as

$$\mathfrak{A} = \frac{(-1)^n\,\mathrm{e}^n}{12(61n-119)}(\mathfrak{A}_1)^{n-1}\mathfrak{A}_2.$$

Consequently, this condition reduces to either $\mathfrak{A}_1 = 0$, that is

$$\varpi = \varpi_{(1)} := \frac{[nk^2 + 2(n+1)\mathfrak{d}]\mathrm{i}}{2(n+1)}, \tag{5.11}$$

or $\mathfrak{A}_2 = 0$, that is

$$\varpi = \varpi_{(\pm)} := \begin{cases} \dfrac{\mathrm{i}}{2}\left[\mathfrak{d} + \dfrac{1}{3}k^2 \pm \sqrt{\left(\mathfrak{d} + \dfrac{1}{3}k^2\right)^2 - 2k^2}\right] & \text{if} \quad n = 2, \\ \dfrac{\mathrm{i}}{2}\left[\mathfrak{d} + \dfrac{1}{2}k^2 \pm \sqrt{\left(\mathfrak{d} + \dfrac{1}{2}k^2\right)^2 - \dfrac{4}{3}k^2}\right] & \text{if} \quad n = 3. \end{cases} \tag{5.12}$$

The dispersion relation (5.11) corresponds to the solution of the perturbation equations which is referred to as a transverse mode, since the amplitudes of the perturbation fields are such that $\delta X = 0$ and $\xi_l \delta X^l = 0$. There is no oscillation in time and the transverse mode is purely damped. The perturbations which are governed by the dispersion relations (5.12) are referred to as longitudinal modes, since the amplitudes $(\delta X, \delta X^j)$ satisfy the conditions $\delta X \neq 0$, $\delta X^j = (\xi_l \delta X^l)\xi^j$. With the aid of (5.12), we easily verify that the imaginary parts of $\omega_{(+)}$ and $\omega_{(-)}$ are strictly positive for all strictly positive values of k. The longitudinal modes decay therefore exponentially with time, even though oscillations can appear when $\mathfrak{d} < (5-n)/n$. Here, as a consequence of $\tau_N \ll \tau_R$, we have $\mathfrak{d} \ll 1$. After defining $k_{(\pm)}$ by

$$k_{(\pm)} := \begin{cases} \sqrt{3(3-\mathfrak{d}) \pm 3\sqrt{3(3-2\mathfrak{d})}} & \text{if} \quad n = 2, \\ \sqrt{2(\frac{4}{3}-\mathfrak{d}) \pm \frac{4}{3}\sqrt{2(2-3\mathfrak{d})}} & \text{if} \quad n = 3, \end{cases}$$

the expressions for $\varpi_{(+)}$ and $\varpi_{(-)}$ may concisely be written as

$$\varpi_{(\pm)} := \frac{\mathrm{i}}{2}\left[\mathfrak{d} + \frac{1}{5-n}k^2 \pm \frac{1}{5-n}\sqrt{\left(k^2 - k_{(-)}^2\right)\left(k^2 - k_{(+)}^2\right)}\right]. \tag{5.13}$$

For the short-wavelength ($k \to \infty$) limit, we get from (5.13) approximately

$$\varpi_{(+)} \cong \frac{\mathrm{i}k^2}{5-n}, \qquad \varpi_{(-)} \cong \frac{\mathrm{i}\mathfrak{d}}{2} + \frac{\mathrm{i}}{4(5-n)}\left[k_{(+)}^2 + k_{(-)}^2 + \left(\frac{k_{(+)}^2 - k_{(-)}^2}{2k}\right)^2\right].$$

In the regions $k \leqslant k_{(-)}$ and $k \geqslant k_{(+)}$, equation (5.13) implies $\mathrm{Re}(\varpi_{(\pm)}) = 0$. Thus, $\mathrm{Re}(\varpi_{(\pm)}) \neq 0$ is only possible when

$$k_{(-)} < k < k_{(+)},$$

and (5.13) leads to

$$\mathrm{Re}(\varpi_{(\pm)}) = \mp\frac{1}{2(5-n)}\sqrt{\left(k^2 - k_{(-)}^2\right)\left(k_{(+)}^2 - k^2\right)}.$$

The longitudinal modes are weakly damped at $k \cong k_{(-)}$ and strongly damped at $k \cong k_{(+)}$.

We now proceed to the study of the case $\tau_R = \infty$. In order to simplify the analysis, we restrict attention to the situation where the wave vector is parallel to the background drift velocity. Given definitions (4.4) and (5.10), this means that

$$|\mathbf{v}| \neq 0, \qquad \lambda := \xi_l w^l = \pm 1.$$

Using (5.4) and (5.5), the substitution from (5.6) into the left-hand side of (5.3) then yields

$$\left[\varpi - \frac{vk(n+1)}{n+u}\right]\delta X - \frac{e\lambda k(n+1)(n-u)}{(n+u)^2}w_l\delta X^l = 0,$$

$$\frac{nk(1-u)}{(n+1)(n-u)}\left[1-u-\frac{(n+u)Hvk\mathrm{i}}{3(n-u)}\right]\delta X w^j - e\lambda\left[\varpi - vk - \frac{1}{2}n(1-u)Dk^2\mathrm{i}\right]\delta X^j$$
$$- e\lambda k\left[\frac{(1-u)(3n+u)v}{n^2-u^2} + \frac{1}{2}n(1-u)Dk\mathrm{i} - \frac{n(n+3u)Hk\mathrm{i}}{3(n-u)^2}\right]w_l\delta X^l w^j = 0,$$

where

$$\varpi := \tau_N\omega, \qquad v := \lambda|\mathbf{v}|, \qquad k := c\tau_N|\mathbf{k}|.$$

Here, of course, the coefficients D and H are defined by (4.6). If the abbreviations

$$\mathfrak{A}_1 := \varpi - vk - \tfrac{1}{2}n(1-u)Dk^2\mathrm{i},$$
$$\mathfrak{A}_2 := (n-u)^2\varpi^2 - \left[2(n-1)(n-u)vk + \tfrac{1}{3}n(n+3u)Hk^2\mathrm{i}\right]\varpi$$
$$- (1-nu)(n-u)k^2 + \tfrac{1}{3}n(n+2+u)Hvk^3\mathrm{i}$$

are introduced, condition (5.8) takes the form

$$\mathfrak{A} = (-\lambda)^n e^n (\mathfrak{A}_1)^{n-1}\mathfrak{A}_2.$$

From $\mathfrak{A}_1 = 0$ we have

$$\varpi = \varpi_{(1)} := vk + \tfrac{1}{2}n(1-u)Dk^2\mathrm{i} \tag{5.14}$$

and from $\mathfrak{A}_2 = 0$ we obtain

$$\mathrm{Re}(\varpi) = \varpi_{r(\pm)} := \frac{k}{(n-u)^2}\left[(n-1)(n-u)v \mp \frac{1}{4}\sqrt{2\sqrt{\mathbb{A}^2+\mathbb{B}^2} - 2\mathbb{A}}\right], \tag{5.15a}$$

$$\mathrm{Im}(\varpi) = \varpi_{i(\pm)} := \frac{k}{2(n-u)^2}\left[\frac{1}{3}n(n+3u)Hk \pm \frac{\mathbb{B}}{\sqrt{2\sqrt{\mathbb{A}^2+\mathbb{B}^2} - 2\mathbb{A}}}\right], \tag{5.15b}$$

where

$$\mathbb{A} := \tfrac{1}{9}n^2(n+3u)^2H^2k^2 - 4n(1-u)^2(n-u)^2,$$
$$\mathbb{B} := \tfrac{4}{3}n(1-u)(3n+u)(n-u)Hvk.$$

The dispersion relation (5.14) yields the transverse mode solution of system (5.9),

$$\delta X = 0, \qquad \xi_l \delta X^l = \lambda w_l \delta X^l = 0,$$

whereas the dispersion relations constructed from equations (5.15*a*) and (5.15*b*) give the longitudinal mode solutions of this system,

$$\delta X \neq 0, \qquad \delta X^j = (\xi_l \delta X^l)\xi^j = \left(w_l \delta X^l\right) w^j.$$

Stability requires the imaginary part of the frequency to be non-negative. For the transverse mode, the stability condition $\mathrm{Im}(\varpi_{(1)}) \geqslant 0$ follows readily from $(4.7)_2$ (i.e., $D > 0$) and (5.14). For the longitudinal modes, it is enough to check that the quantity defined by

$$Z := \tfrac{4}{81}\,[n(n+3u)Hk]^4(\mathbb{A}^2 + \mathbb{B}^2) - \left[\mathbb{B}^2 + \tfrac{2}{9}n^2\mathbb{A}(n+3u)^2H^2k^2\right]^2$$

satisfies the inequality $Z \geqslant 0$, since the stability conditions $\varpi_{i(+)} \geqslant 0$ and $\varpi_{i(-)} \geqslant 0$ are direct consequences of $(4.7)_3$ and this inequality:

$$(H > 0, Z \geqslant 0) \Rightarrow \frac{1}{3}n(n+3u)Hk \pm \frac{\mathbb{B}}{\sqrt{2\sqrt{\mathbb{A}^2+\mathbb{B}^2} - 2\mathbb{A}}} \geqslant 0.$$

The key to the proof of $Z \geqslant 0$ consists of using the identity

$$Z = (3n+u)^2(n-u)^7u\left[\tfrac{4}{3}n(1-u)Hk\right]^4$$

and the fact that $n > 1$ and $0 \leqslant u < 1$.

Now, given the dispersion relations for the longitudinal modes, equations (5.15*a*) and (5.15*b*), we find the two longitudinal-mode 'frequencies' in the small k-limit have the values

$$\varpi_{(\pm)} = \frac{k}{n-u}[(n-1)v \mp \sqrt{n}(1-u)] + \frac{\sqrt{n}(\sqrt{n} \pm v)Hk^2\mathrm{i}}{6(\sqrt{n} \mp v)^2}.$$

For the short-wavelength ($k \to \infty$) limit, assuming for the moment that $v \neq 0$, the leading terms in the longitudinal dispersion relations are of the form

$$\varpi_{r(\pm)} = \frac{k}{n-u}\left[(n-1)v \mp \frac{(1-u)(3n+u)|v|}{n+3u}\right],$$

$$\varpi_{i(+)} = \begin{cases} \dfrac{n(n+3u)Hk^2}{3(n-u)^2} & \text{if} \quad v > 0, \\[2ex] \dfrac{3(1-u)^2}{(n+3u)H}\left[1 + \dfrac{9(1-u)^2(n-u)^2}{n(n+3u)^2H^2k^2}\right] & \text{if} \quad v < 0, \end{cases}$$

$$\varpi_{i(-)} = \begin{cases} \dfrac{3(1-u)^2}{(n+3u)H}\left[1 + \dfrac{9(1-u)^2(n-u)^2}{n(n+3u)^2H^2k^2}\right] & \text{if} \quad v > 0, \\[2ex] \dfrac{n(n+3u)Hk^2}{3(n-u)^2} & \text{if} \quad v < 0. \end{cases}$$

Finally, using the identity

$$\frac{\mathbb{B}}{\sqrt{2\sqrt{\mathbb{A}^2+\mathbb{B}^2} - 2\mathbb{A}}} = \begin{cases} \frac{1}{2}\sqrt{2\sqrt{\mathbb{A}^2+\mathbb{B}^2} - 2\mathbb{A}} & \text{if} \quad v > 0, \\ -\frac{1}{2}\sqrt{2\sqrt{\mathbb{A}^2+\mathbb{B}^2} - 2\mathbb{A}} & \text{if} \quad v < 0, \end{cases}$$

we verify that equations (5.15*a*) and (5.15*b*) reduce to the dispersion relations (5.12) with $\mathfrak{d} = 0$ as v approaches zero from either the left ($\lambda = -1$) or the right ($\lambda = 1$). Similar remarks apply also to the dispersion relations for the transverse modes, equations (5.11) and (5.14).

6. Final remarks

In this paper, we derived a set of nonlinear second-order parabolic equations for describing the motion of a phonon gas in the regime where normal processes substantially dominate over resistive ones. The derivation was based on a Chapman–Enskog-like perturbative expansion of the phase density about a displaced Planck distribution. It should be re-iterated that the main advantage of using the displaced Planck distribution is that the drift velocity is incorporated into the model in a non-perturbative manner, thereby allowing virtually arbitrarily large values for the individual components of the heat flux. The proposed model is, in essence, a parabolization of the Nielsen–Shklovskii hyperbolic model and is effectively obtained by the addition of an appropriate combination of the spatial derivative terms to the right-hand side of

$$M^{ij} = \frac{(n+1)c^2 e}{n+u} v^{\langle i} v^{j\rangle}.$$

Assuming that the higher-order terms of expansion (2.11) represent a decreasing series of perturbations, one can truncate expansion (2.11) to some low order. The zeroth-order truncation gives the hyperbolic system comprised of equations (2.20) and (2.22). This system is very similar to the system of Nielsen and Shklovskii, but differs from it in that the heat flux balance equation does not contain the source term due to resistive processes. Of course, such a source term vanishes if $\tau_R = \infty$. Here, however, $\tau_R \neq \infty$ and the missing non-zero source term is recovered when the first-order truncation is considered. In principle, employing equation (2.5*b*), system (3.7) can be formulated in terms of the energy density and the heat flux as system (3.6). Nevertheless, the main disadvantage in the use of this formulation is that an explicit expression of equations (3.2) and (3.3) in terms of the energy density and the heat flux is much more complicated than it was for the formulation in terms of (e, v^i). The use of (e, v^i) is thus evidently preferable to the use of (e, q^i) in the treatment of nonlinear heat flux problems.

The equations derived by means of the first-order approximation are linearly stable at all wavelengths and they yield results consistent with the second law of thermodynamics. In order to prove that these equations are indeed entropy consistent, the entropy balance law was constructed from macroscopic thermodynamic theory using the Nielsen–Shklovskii entropy function and the energy and heat flux balance equations. The same entropy balance law follows from kinetic theory using Boltzmann's H-theorem in conjunction with the Chapman–Enskog expansion. In both cases it is found that the irreversible entropy production is non-negative, in agreement with the second law of thermodynamics. If $\tau_N = 0$ and $\tau_R = \infty$, the entropy production is zero and we obtain the results identical to, and consistent with, those derived from the zeroth-order approximation. The second- and higher-order approximations were not discussed, however, because the equations associated with these approximations usually violate the second law of thermodynamics and exhibit instability to small wavelength disturbances. Analogous problems are encountered when considering the Chapman–Enskog expansion method as applied to classical gases. There, the results of the computations show that under certain flow conditions, the conventional Burnett equations are not consistent with thermodynamics, while the Euler and Navier–Stokes equations provide entropy consistent results [18, 19].

Perhaps a word should be said about one possible modification of the present method. In the regime where τ_N is small and τ_R is large, another way of deriving the equations of phonon hydrodynamics is based on the use of the two small parameters which are inserted into Callaway's model as follows:

$$\partial_t f + c g^i \partial_i f = \frac{\varepsilon_1}{\tau_R}(F - f) + \frac{1}{\varepsilon_2 \tau_N}(F_d - f). \tag{6.1}$$

Setting

$$f = \sum_{l=0}^{\infty} \sum_{m=0}^{l} \epsilon_1^{l-m} \epsilon_2^{m} f_{l(m)}, \tag{6.2}$$

the next step is to look for solutions of equation (6.1) such that the zeroth-order distribution $f_{0(0)}$ is the displaced Planck distribution and the first- and higher-order terms are chosen so they have no contribution to the moments expressed in equations (2.4*a*) and (2.4*b*). The zeroth-order truncation of expansion (6.2) gives the hyperbolic equations of section 2.2. The first-order truncation formally requires a knowledge of the coefficients $f_{1(0)}$ and $f_{1(1)}$ and can be shown to give the parabolic equations of section 3 when the fact that $f_{1(0)}$ equals zero is used. Consequently, the conclusion is that although this modification of the Chapman–Enskog method more clearly emphasizes the difference between resistive and normal processes, it yields the results which practically do not differ from those obtained by inserting only one small parameter into Callaway's model.

The stationary flow equations were not investigated and the related issue of surface boundary conditions was not addressed. On the understanding that N^{ij} is defined by (3.2), equation (3.7*b*) contains the second-order spatial derivatives of the hydrodynamic variables. Therefore, as compared to the first-order hyperbolic system of Nielsen and Shklovskii, additional boundary conditions are needed for the solution to the stationary flow equations to be uniquely determined; different solutions can result based on the choice of boundary values. Since our primary aim was to formulate the model and to demonstrate that the equations of this model are linearly stable at all wavelengths and give results consistent with the second law of thermodynamics, the problem of supplying boundary conditions to match a set of equations will be discussed in a separate paper.

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References

[1] Nielsen H and Shklovskii B I 1969 Heat transfer and second sound in dielectrics with large drift velocities of a phonon gas *Zh. Eksp. Teor. Fiz.* **56** 709–17

[2] Hardy R J 1970 Phonon Boltzmann equation and second sound in solids *Phys. Rev.* B **2** 1193–207

[3] Peierls R E 1955 *Quantum Theory of Solids* (London: Oxford University Press)

[4] Gurevich V L 1980 *Kinetika Fononnykh Sistem* (Moscow: Nauka)

[5] Alascio B, Nazareno H and Carabelli G 1968 Effect of normal processes on the thermal conductivity of finite insulating samples *Nuovo Cimento* B **55** 362–70

[6] Beck H 1975 Second sound and related thermal conduction phenomena *Dynamical Properties of Solids* vol 2, ed G K Horton and A A Maradudin (Amsterdam: North-Holland) pp 205–84

[7] Larecki W and Piekarski S 1991 Symmetric conservative form of low-temperature phonon gas hydrodynamics: I. Kinetic aspects of the theory *Nuovo Cimento* D **13** 31–53

[8] Larecki W 1992 Symmetric conservative form of low-temperature phonon gas hydrodynamics: II. Equations of heat transport and thermal waves in the case of linear isotropic approximation of phonon frequency spectrum *Nuovo Cimento* D **14** 141–76

[9] Sussmann J A and Thellung A 1963 Thermal conductivity of perfect dielectric crystals in the absence of Umklapp processes *Proc. Phys. Soc.* **81** 1122–30

[10] Nielsen H and Shklovskii B I 1968 Nonlinear heat transfer in dielectrics in the regime of viscous phonon drift *Fiz. Tverd. Tela* **10** 3602–7

[11] Karlin I V, Dukek G and Nonnenmacher T F 1997 Gradient expansions in kinetic theory of phonons *Phys. Rev.* B **55** 6324–9

[12] Zakharchenko P A and Radkevich E V 2004 Central manifold and the problems of the Chapman–Enskog projection for the Boltzmann–Peierls equation *Dokl. Phys.* **49** 451–5

[13] Palin V V and Radkevich E V 2006 Navier–Stokes approximation and problems of the Chapman–Enskog projection for kinetic equations *J. Math. Sci.* **135** 2721–48

[14] Callaway J 1959 Model for lattice thermal conductivity at low temperatures *Phys. Rev.* **113** 1046–51

[15] Callaway J 1991 *Quantum Theory of the Solid State* (Boston: Academic)

[16] Kawashima S and Shizuta Y 1988 On the normal form of the symmetric hyperbolic–parabolic systems associated with the conservation laws *Tôhoku Math. J.* **40** 449–64

[17] Kawashima S and Yong W A 2004 Dissipative structure and entropy for hyperbolic systems of balance laws *Arch. Ration. Mech. Anal.* **174** 345–64

[18] Comeaux K A, Chapman D R and MacCormack R W 1995 An analysis of the Burnett equations based in the second law of thermodynamics *AIAA Paper No 95-0415*

[19] Agarwal R K, Yun K Y and Balakrishnan R 2001 Beyond Navier–Stokes: Burnett equations for flows in the continuum-transition regime *Phys. Fluids* **13** 3061–85